A Topology for Gluing Differential Bundles

A Zariski Topology for Tangent Categories

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Based on Joint Work with JS Lemay

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2 Motivation, Background, and Schemes, Oh My!

3 Zariski Opens and Quasi-Coherent Sheaves

4 Differential Bundles and Modules

5 Tangentifying Zariski and fpqc Topologies

Why This is Cool

Theorem

Given an affine tangent category $\mathscr{C}$, there are Zariski and fpqc Grothendieck topologies J_{Zar} and J_{fpqc} on $\mathscr{C}$ defined in terms of differential bundles.

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Theorem

Given an affine tangent category $\mathscr{C}$, there are Zariski and fpqc Grothendieck topologies J_{Zar} and J_{fpqc} on $\mathscr{C}$ defined in terms of differential bundles.
When $\mathscr{C} = \mathbf{Cring}^{\text{op}}$, J_{Zar} and J_{fpqc} coincide with the Zariski and fpqc topologies.

Why This is Cool

The Take-Away

The sheaf topos $\mathbf{Shv}(\mathscr{C}, J_{\mathrm{Zar}})$ lets us talk about a category where we can glue differential bundles and, by carefully choosing a subcategory of $\mathbf{Shv}(\mathscr{C}, J_{\mathrm{Zar}})$[a], this in turn gives us a way to define what it means to be a scheme in a tangent category!

[a] There are details that go into this which remain to be checked carefully.

Why This is Cool

An Important Question

The question I hope to answer for you all today:

How do we glue differential bundles on, if not all tangent categories, those which are sufficiently "affine scheme-y" with "module-like" differential bundles? What do "sufficiently affine scheme-y" and "module-like" mean?

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Algebraic Geometry Background

The Reconstruction Theorem (Gabriel, Rosenberg)

Let X and Y be quasi-separated schemes and let $\mathbf{QCoh}(X)$ and $\mathbf{QCoh}(Y)$ denote their categories of quasi-coherent sheaves. If $\mathbf{QCoh}(X) \simeq \mathbf{QCoh}(Y)$ then $X \cong Y$.

Algebraic Geometry Background

But Why Quasi-Coherent Sheaves?

For today, take this as a reason to care about/define quasi-coherent sheaves: When $X \cong \operatorname{Spec} A$ is an affine scheme,

$$\mathbf{QCoh}(X) \simeq A\text{-}\mathbf{Mod}\,.$$

So $\mathbf{QCoh}(X)$ is the category of scheme modules which are given by gluing modules over the affine patches of X!

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But What is a Quasi-Coherent Sheaf Really?

The full definition of a $\mathbf{QCoh}(X)$ is as the full subcategory of $\mathcal{O}_X$-$\mathbf{Mod}$ generated by sheaves on X which are locally cokernels of morphisms of free sheaves of $\mathcal{O}_{U_i}$-modules.
Is there a different/more direct/tangent-theoretic to describe $\mathbf{QCoh}(X)$?

And Now, a Tangent

In [1], G. Cruttwell and J.-S. Lemay showed that for the tangent category **Sch** of schemes, for any scheme X there is an opposite equivalence

$$\mathbf{DBun}(X) \simeq \mathbf{QCoh}(X)^{\mathrm{op}},$$

where $\mathbf{DBun}(X)$ denotes the category of differential bundles over X. So when $\mathscr{C} = \mathbf{Sch}$, we can glue differential bundles by working Zariski locally in X. But what should "Zariski locally" mean in a tangent category?

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What Does Being Zariski Open Buy You?

We want to give a short description of what it means to be a Zariski open

$$f : V \to U$$

in **AffSch**. Here are some observations from algebraic geometry that help us describe this.

What Does Being Zariski Open Buy You?

A Zariski open $V \to U$ must be described by an open immersion $\gamma : V \to U$ of schemes. This means f satisfies three properties right away:

1. It is flat (so the pullback $f^* : \mathbf{QCoh}(U) \to \mathbf{QCoh}(V)$ is left exact).
2. It is monic in **AffSch**.
3. It is finitely presented.

Characterizing Zariski Opens in Affine Schemes

What Does Being Zariski Open Buy You?

By saying that a morphism $f : V \to U$ is finitely presented, because we are working with $\mathbf{AffSch} \simeq \mathbf{Cring}^{\mathrm{op}}$ we mean in the usual algebraic sense by [EGA 4, Proposition 8.14.2]: for any filtered diagram $D : I \to \mathbf{AffSch}_{/U}$, the natural map

$$\varinjlim_{i \in I} \mathbf{AffSch}_{/U}(Di, V) \cong \mathbf{AffSch}_{/U}\left(\varprojlim_{i \in I} Di, V\right)$$

is an isomorphism.

Characterizing Zariski Opens in Affine Schemes

ACHTUNG

Because scheme theory, it is worth noting that there is some magic that happens because the schemes V and U are *affine.* In general, a morphism $f : X \to Y$ of schemes is *locally finitely presented* (cf. [EGA 4]) if for any filtered diagram $D : I \to \mathbf{Sch}_{/Y}$ the map

$$\varinjlim_{i \in I} \mathbf{Sch}_{/Y}(Di, X) \cong \mathbf{Sch}_{/Y}\left(\varprojlim_{i \in I} Di, X\right)$$

is an isomorphism. To get full on finite presentation, we need f to *also* be quasi-compact and quasi-separated. However, this is not a problem for affine schemes!

Characterizing Zariski Opens in Affine Schemes

Going the Other Way

So if we have an arbitrary map $f : X \to Y$ of affine schemes, when is it a Zariski open? That is, is it enough to assume the three points (being flat, being monic, and being of finite presentation) given above?

Characterizing Zariski Opens in Affine Schemes

Going the Other Way

Let's assume that we have a flat and finitely presented morphism $f : X \to Y$ of affine schemes. Then a technical lemma [Stacks Project, Tag 00I1] shows that f is an open morphism.

The only thing we're missing is that we need to know that $f : X \to Y$ is monic, but this is not automatic. We need to ask for this!

Going the Other Way

So if we ask for $f : X \to Y$ to be flat, finitely presented, and monic we know that f is necessarily an open immersion and hence a Zariski open in **AffSch**! Equivalently, the affine Zariski opens $\operatorname{Spec} B \to \operatorname{Spec} A$ are exactly the morphisms which are:

1. flat;
2. monic;
3. finitely presented.

Characterizing Zariski Opens in Affine Schemes

Going the Other Way

Since we know Zariski opens now, once we explain what it means for differential bundles to be sufficiently module-like, we'll be able to start talking a Zariski topology on tangent categories.

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An Important (Relative) Approach

In [2], B. Toën and M. Vaquiè took an approach towards extending the notion of a scheme to a category of commutative monoids in a complete and cocomplete symmetric monoidal closed category equipped with a sufficiently "module-like" (quasi-coherent) pseudofunctor.

Module-Like Differential Bundles

An Important (Relative) Approach

Sufficiently module-like means: Start with a finitely complete category $\mathscr{C}$ and a pseudofunctor $M : \mathscr{C}^{\mathsf{op}} \to \mathfrak{Cat}$ where:

- For any object X of $\mathscr{C}_0$, $M(X)$ is complete and cocomplete.

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Sufficiently module-like means: Start with a finitely complete category $\mathscr{C}$ and a pseudofunctor $M : \mathscr{C}^{\mathrm{op}} \to \mathfrak{Cat}$ where:

- For any object X of $\mathscr{C}_0$, $M(X)$ is complete and cocomplete.
- For any morphism $f : X \to Y$ in $\mathscr{C}$, the functor $M(f) : M(Y) \to M(X)$ possesses a right adjoint $R(f) : M(X) \to M(Y)$ which is isomorphism-reflecting and pseudofunctorial in $\mathscr{C}$.

Module-Like Differential Bundles

An Important (Relative) Approach

- For all pullback diagrams

$$\begin{array}{ccc} W & \xrightarrow{t} & X \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

in $\mathscr{C}$, the corresponding mate of the natural isomorphism

$$\begin{array}{ccc} M(Z) & \xrightarrow{M(f)} & M(X) \\ {\scriptstyle M(g)}\downarrow & \Downarrow{\scriptstyle \rho} & \downarrow{\scriptstyle M(t)} \\ M(Y) & \xrightarrow[M(s)]{} & M(W) \end{array}$$

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Module-Like Differential Bundles

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- $$\begin{array}{ccc} M(Z) & \xrightarrow{M(f)} & M(X) \\ {\scriptstyle M(g)}\downarrow & \Downarrow \rho & \downarrow {\scriptstyle M(t)} \\ M(Y) & \xrightarrow[M(s)]{} & M(W) \end{array}$$

  $$\beta : M(g) \circ R(f) \xRightarrow{\cong} R(s) \circ M(t)$$

  or, more suggestively,

  $$\beta : g^* \circ f_* \xRightarrow{\cong} s_* \circ t^*.$$

Module-Like Differential Bundles

Why Isomophism Reflecting?

Here is an important remark justifying needing the adjoint f_* of f^* to be isomorphism-reflecting.

Module-Like Differential Bundles

Why Isomophism Reflecting?

Given a map $f : \operatorname{Spec} B \to \operatorname{Spec} A$, the adjoint

$$\mathbf{QCoh}(\operatorname{Spec} A) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathbf{QCoh}(\operatorname{Spec} B) \qquad (f^* \dashv f_*)$$

corresponds to the adjunction:

$$A\text{-}\mathbf{Mod} \underset{(-)\otimes_A B}{\overset{\operatorname{Res}}{\rightleftarrows}} B\text{-}\mathbf{Mod} \qquad ((-)\otimes_A B \dashv \operatorname{Res})$$

Why Isomophism Reflecting?

In the adjoint

$$A\text{-}\mathbf{Mod} \underset{(-)\otimes_A B}{\overset{\mathrm{Res}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} B\text{-}\mathbf{Mod}$$

Res is isomorphism reflecting.

Module-Like Differential Bundles

Our Take-Away

We will need to know that **DBun** is sufficiently module like by having a pseudofunctor taking values $\mathbf{DBun} : \mathscr{C}^{\mathsf{op}} \to \mathsf{Adj}$ with isomorphism-reflecting adjoints to the pullback functors f^*.

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Our Goal

We now want to describe how to combine the two main ingredients which go into the relative algebraic geometry construction can be tangentified in order to construct a Zariski and fpqc topology on a tangent category.

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An Important Remark

Recall that [1] showed that for a scheme X,

$$\mathbf{DBun}(X) \simeq \mathbf{QCoh}(X)^{\mathrm{op}}.$$

As such, if we want to work with the differential bundle technology we need to flip the arrows of literally everything in sight and be really, really careful.

Tangent Categories for Algebraic Geometry

Algebraic Geometry Tangent Categories

An affine tangent category $\mathscr{C}$ is a tangent category which satisfies the following assumptions:

- The differential bundle pseudofunctor takes values in the bicategory Adj whose objects are categories, whose morphisms $f : \mathscr{C} \to \mathscr{D}$ are adjunctions

  $$\mathscr{C} \underset{f_R}{\overset{f^L}{\leftrightarrows}} \mathscr{D} \quad (\perp)$$

  and whose 2-cells $\alpha : f \Rightarrow g$ are natural transformations $\alpha : f \Rightarrow g$:

  $$\mathbf{DBun} : \mathscr{C}^{\mathrm{op}} \to \mathrm{Adj}$$

Tangent Categories for Algebraic Geometry

Algebraic Geometry Tangent Categories

- The right adjoint component of $\mathbf{DBun} : \mathscr{C}^{\mathrm{op}} \to \mathrm{Adj}$ is a pseudofunctor $\mathbf{DBun} : \mathscr{C}^{\mathrm{op}} \to \mathfrak{Cat}$ where for any morphism $f : X \to Y$ in $\mathscr{C}$, the functor $f^* : \mathbf{DBun}(Y) \to \mathbf{DBun}(X)$ acts via pullback, i.e., via the formula (induced by)

$$f^*(q : E \to Y) := \mathrm{pr}_2 : E \times_Y X \longrightarrow X$$

  for all differential Y-bundles E.
- The left adjoint $f_!$ of f^* is isomorphism-reflecting for all morphisms f of $\mathscr{C}$.

Tangent Categories for Algebraic Geometry

Algebraic Geometry Tangent Categories

- Given a pullback diagram

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in $\mathscr{C}$, the mate of the natural isomorphism

$$\begin{array}{ccc} \mathbf{DBun}(Z) & \xrightarrow{f^*} & \mathbf{DBun}(X) \\ {\scriptstyle g^*}\downarrow & \Downarrow \rho & \downarrow{\scriptstyle t^*} \\ \mathbf{DBun}(Y) & \xrightarrow[s^*]{} & \mathbf{DBun}(W) \end{array}$$

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- the mate of the natural isomorphism

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is a natural isomorphism

$$f^* \circ g_! \overset{\cong}{\Longrightarrow} t_! \circ s^*.$$

Examples of affine Tangent Categories

Here are three nice examples of affine tangent categories.

- The category $\mathbf{AffSch}_{/A} = \mathbf{AffSch} \downarrow \operatorname{Spec} A$ for any commutative ring A.
- The CDC $\mathbb{R}\,\mathbf{Smooth}$ of smooth maps between finite dimensional Euclidean spaces.
- For any commutative rig C, the CDC $A\mathbf{Poly}$ of polynomials in finitely many variables with coefficients in A.

A Non-Example of an affine Tangent Category

The category **SMan** is not an affine tangent category! This is because for any smooth manifold M, $\mathbf{DBun}(M)$ is equivalent to the category of vector bundles on M and in general the pullback functor $f^* : \mathbf{DBun}(N) \to \mathbf{DBun}(M)$ need not have an isomorphism-reflecting left adjoint.

A Non-Example of an affine Tangent Category

The category **Sch** is not an affine tangent category! This is because in general if we have a pullback square

$$\begin{array}{ccc} W & \xrightarrow{t} & X \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

the base change

$$\beta : f^* \circ g_! \Rightarrow t_! \circ s^*$$

need not be an isomorphism. It is when g is affine or when f is flat and g is quasi-compact quasi-separated, however.

Tangent Categories for Algebraic Geometry

Theorem/A Recipe for Building More affine Tangent Categories

Let $\mathscr{C}$ be an affine tangent category. Then $\mathrm{Ind}(\mathscr{C})$ is an affine tangent category.

Corollary

The category $\mathrm{Ind}(\mathbf{AffSch})$ of affine formal schemes, the category $\mathrm{Ind}(\mathbb{R}\,\mathbf{Smooth})$ of ind-Euclidean spaces, and the category $\mathrm{Ind}(A\mathbf{Poly})$ of ind-polynomials are all affine categories.

Flat Morphisms in a Tangent Category

Let $\mathscr{C}$ be an affine tangent category. We say that a morphism $f : X \to Y$ is flat if the functor

$$f^* : \mathbf{DBun}(Y) \to \mathbf{DBun}(X)$$

is right exact.

The Zariski and fpqc Topologies

Finitely Presented Morphisms in a Tangent Category

Let $\mathscr{C}$ be an affine tangent category. We say that a morphism $f : X \to Y$ is finitely presented if for all cofiltered diagrams $D : I \to (\mathscr{C} \downarrow Y)$ admitting a limit there is an isomorphism

$$(\mathscr{C} \downarrow Y)\left(\varprojlim_{i \in I} D(i), X\right) \cong \varinjlim_{i \in I} (\mathscr{C} \downarrow Y)\,(D(i), X)\,.$$

The Zariski and fpqc Topologies

Zariski Opens in a Tangent Category

Let $\mathscr{C}$ be an affine tangent category. We say that a morphism $f : X \to Y$ is a Zariski open if f is monic, flat, and finitely presented.

Quasi-Compact Covers in a Tangent Category

Let $\mathscr{C}$ be an affine tangent category. Then we say that a collection of morphisms $C = \{f_i : X_i \to X \mid i \in I\}$ is a cover in $\mathscr{C}$ if the functor

$$\mathbf{DBun}(X) \xrightarrow{\langle f_i^* \rangle_{i \in I}} \prod_{i \in I} \mathbf{DBun}(X_i)$$

is isomorphism-reflecting. Additionally, we say that C is quasi-compact if it admits a finite refinement.

The Zariski and fpqc Topologies

The Covers and Pretopologies

Let $\mathscr{C}$ be an affine tangent category and let X be an object of $\mathscr{C}$ with $C = \{f_i : X_i \to X \mid i \in I\}$ is a cover of X in $\mathscr{C}$. We say:

- C is a Zariski cover of X if C is quasi-compact and if f_i is a Zariski open for all $i \in I$.
- C is an fpqc cover of X if C is quasi-compact and if f_i is flat for all $i \in I$.

Write τ_{Zar} for the collection of all Zariski covers in $\mathscr{C}$ and τ_{fpqc} for the collection of all fpqc covers in $\mathscr{C}$.

The Zariski and fpqc Topologies

Theorem

Let $\mathscr{C}$ be an affine tangent category. The pretopologies τ_{Zar} and τ_{fpqc} both induce Grothendieck topologies on $\mathscr{C}$.

Proposition

Let A be a commutative ring and let $\mathscr{C} := \mathbf{AffSch}_{/A}$. Then if J_{Zar} and J_{fpqc} are the tangent Zariski and fpqc topologies from algebraic geometry on $\mathbf{AffSch}_{/A}$, these topologies coincide with the classical Zariski and fpqc topologies.

The Zariski and fpqc Topologies

To Do Still

Here is a list of stuff we'll be working on and sorting out:

- It should be the case that **DBun** is an fpqc stack for any affine tangent category. If true, this means not only can we glue differential bundles, we can even do flat gluing of differential bundles provided we keep track of the glue we used!

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- We need to prove that the category of tangent schemes is equivalent at least to the usual category of schemes.
- Check for the difference between tangent schemes in $\operatorname{Ind}(\mathscr{C})$ for an affine tangent category $\mathscr{C}$ and $\operatorname{Ind}(\mathbf{Sch}_{\mathscr{C}})$. In other words, are formal schemes the gluing of affine formal schemes?

The Zariski and fpqc Topologies

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- Check for the difference between tangent schemes in $\operatorname{Ind}(\mathscr{C})$ for an affine tangent category $\mathscr{C}$ and $\operatorname{Ind}(\mathbf{Sch}_{\mathscr{C}})$. In other words, are formal schemes the gluing of affine formal schemes?
- Figure out if there's a way to get **SMan** into this story...

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The Last Slide

The End

Thanks for coming and listening everybody!